Functional Analysis 2022

MAST90020

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Contents

Statements and lemmas in a blue box are intended to be exercises

1 Metric and Banach Spaces Reviewed

Definition 1

A metric space is a set and a function (X, d) such that the function satisfies: $\forall x, y, z \in X$

- $d : X \times X \to \mathbb{R}$
- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Prove that the euclidean metric is a metric as well as the sup metric on continuous functions and d1 metric on L1s

Definition 2

A sequence $(x_n)_{n\geq 1}$ in (X, d) converges to $x \in X$

$$\iff lim_{n\to\infty} x_n = x$$

$$\iff \forall \epsilon > 0 \exists N > 0 \forall n > N, d(x_n, x) < \epsilon$$

Definition 3

A sequence $(x_n)_{n\geq 1}$ in (X, d) is Cauchy

$$\iff \forall \epsilon > 0 \exists N > 0 \forall n, m > N, d(x_n, x_m) < \epsilon$$

Lemma 1

Convergence of a sequence $\implies$ the sequence is Cauchy
(The reverse implication holds only in a complete metric space)

The quintessential metric space is of course $\mathbb{R}^n$ with the euclidean distance. For this course the key example of a metric space is $cts([0, 1], \mathbb{R}) = \{f : [0, 1] \to \mathbb{R} | f \text{ is continuous}\}$. We can put the following metrics on this set

- $d_\infty(f, g) = sup_{x\in[0,1]} |fx - gx|$ (because $[0, 1]$ is compact and $\mathbb{R}$ is metrisable)
- $d_p(f, g) = (\int_{[0,1]} |fx - gx|^p dx)^{\frac{1}{p}}$ (because $[0, 1]$ is compact Hausdorff and $\mathbb{R}$ is a field)

Definition 4

Two metrics on the same space$X$ are equivilent (denoted $d_1 \sim d_2$)

$$\iff \exists C > 0 \forall x, y \in X, \frac{1}{C} d_2(x, y) < d_1(x, y) < C d_2(x, y)$$

There is a strict one way implication $(f_n) \xrightarrow{d_\infty} f \implies (f_n) \xrightarrow{d_1} f$. This can be seen by considering the simple counterexample of the triangle of width $\frac{1}{n}$ as our sequence of functions. This tells us that $d_\infty \not\sim d_1$.

Lemma 2

Two equivilent metrics generate the same topologies

Lemma 3

If two metrics are equivilent then a sequence converges in one if and only if it converges in the other.

For two equivilent metrics a sequence is cauchy in one iff it is cauchy in the other

Definition 5

A metric space (X, d) is complete

$$\iff [x_n \to x \iff (x_n) \text{ is Cauchy}]$$

Definition 6

A function $f : X \to Y$ is continuous $\iff \forall x \in X \forall \epsilon > 0 \exists \delta \forall x' \in X, d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$

Definition 7

A bijective distance preserving map between metric spaces is an isometry

1.1 Metric Space Completion

For any metric space (X, d) there is a complete metric space $(\tilde{X}, \tilde{d})$ and an isometric map (isometry onto its image) $\iota : X \to \tilde{X}$ such that ιX is dense inside of $\tilde{X}$.

The map $\tilde{d}$ is a well defined metric and $(\tilde{X}, \tilde{d})$ is a complete metric space

1.2 Basic Topology

Definition 8

The open ball around y of radius r is $B(y, r) = B_r(y) = \{x \in X | d(x, y) < r\}$

Definition 9

A set $\mathcal{U}$ is open in a metric topology $\iff \forall y \in \mathcal{U} \exists r > 0, B_r(y) \subseteq \mathcal{U}$

Definition 10

The closed ball around y of radius r is $\bar{B}(y, r) = \bar{B}_r(y) = \{x \in X | d(x, y) \leq r\}$

Definition 11

A set $\mathcal{C}$ is closed in a topology $(X, \mathcal{U}) \iff \exists U \in \mathcal{U}, \mathcal{C} = U^C$ (the compiment of an open set)

Lemma 4

A set in a metric space is closed $\iff$ it contains all of its limit points

Definition 12

A topological space is compact $\iff$ every cover has a finite subcover

Definition 13

A metric space is sequentially compact $\iff$ every sequence has a convergent subsequence

Lemma 5

A metric space is sequentially compact $\iff$ The topology induced by the metric is compact

Definition 14

A map in a topological space is continuous iff the preimage of open sets is open.

Lemma 6

A map between metric spaces is continuous iff it is continuous in the topological sense when the metric space is given the metric topology.

Lets apply some of this theory to the example of $C[0,1] = (Cts([0,1], \mathbb{R}), d_\infty)$.
Recall that convergence in the d_∞ metric is equivilent to uniform convergence of functions. We produced the example of the triangle above whereby a sequence of functions in $C[0,1]$ converged pointwise to $f(x) = I(x = \frac{1}{2})$, which is not a continuous function on $[0,1]$. Thus there is a sequence that has no convergent subsequence and we have proven $C[0,1]$ is not sequentially compact. Thus it is not compact. Also note that the closed unit ball in $C[0,1]$ is not compact.

1.3 Normed Linear Spaces

Definition 15

A function from a vector space V to a field (for example $\mathbb{C}$), which we will denote $\|-\| : V \to \mathbb{C}$, is a norm if it satisfies the following $\forall x, y \in V, \forall \lambda \in \mathbb{C}$:

- $\|x\| \geq 0$
- $\|x\| = 0 \iff x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda| \|x\|$

The vector space and function pair, $(V, \|-\|)$, is called a normed linear space (NLS).

Lemma 7

Any norm induces the following metric

$$d(x, y) = \|x - y\|$$

Definition 16

A NLS in which the induced metric is complete is called a Banach space

$C[0,1]$ can be thought of as a normed linear space with the following norm

$$\|f\|_\infty = d_\infty = sup_{x\in[0,1]} |fx|$$

This norm clearly induces the d_∞ metric, moreover this is a Banach space.

Definition 17

A linear transformation $T : V_1 \to V_2$, between two NLS is bounded

$$\iff \exists C > 0 \forall v \in V_1, \|Tv\|_2 \leq C\|v\|_1$$

Theorem 1

For a linear transformation T as above

$$T \text{ bounded } \iff T \text{ continuous } \iff T \text{ continuous at } 0 \in V_1$$

Proof. Continuous at zero $\implies$ continuous everywhere: Take a convergent sequence in V_1 $x_n \to x \implies x_n - x \to 0$. Then by continuity at zero we have that $T(x_n - x) \to 0$ in V_2. By linearity we have that

$$T(x_n - x) = T(x_n) - T(x) \to 0 \implies T(x_n) \to T(x)$$

We are using here that a function is continuous at a point iff it preserves limits at that point.

Continuous $\implies$ bounded: T continuous $\implies T^{-1}(B_1(0))$ (the ball of radius 1 around 0 in V_2) is open, moreover $0 \in T^{-1}(B_1(0))$. Thus $\exists r > 0$ such that $B_r(0) \subseteq T^{-1}(B_1)(0)$. Now we scale all the points in the space into this ball as follows

$$\forall x \in V_1 \setminus \{0\} \quad \|T(\frac{xr}{2\|x\|})\| \leq 1 \iff \|Tx\| \leq \frac{2}{r}\|x\|$$

So for any point in V_1 we scale it into the ball around 0 that exists by continuity and take its image sending it to a ball of radius one around 0 in the codomain by construction. We then use linearity to show that T is bounded.

Bounded $\implies$ Continuous at 0: Let $x_n \to 0$ in V_1 then by boundedness we know that

$$\|Tx_n\| \leq C\|x_n\| \to 0$$

Thus $T(x_n) \to 0$ and again we use that continuity is given iff it preserves limits at a point and so we get that T is continuous at zero. □

Theorem 2

For two NLS $(V_1, \|\cdot\|_1), (V_2, \|\cdot\|_2)$, where V_2 is complete and a linear map $T : V_1 \to V_2$ that is bounded on a subset $D \subseteq V_1$ then there is a unique BLT $\tilde{T} : \tilde{V_1} \to V_2$ from the completion of V_1 that is bounded by the same number and equal on the restriction i.e. $\tilde{T}|_{V_1} = T$

Proof. Take an element $x \in \tilde{V}$ then $\{Tx_n\}$ is Cauchy because

$$\forall \epsilon > 0 \, \exists N \, \forall n, m > N \, \|x_n - x_m\| < \frac{\epsilon}{C} \implies \|T(x_n) - T(x_m)\| = \|T(x_n - x_m)\| \leq C\|x_n - x_m\| < \epsilon$$

But we know that V_2 is complete and so $\exists y \in V_2$ such that $T(x_n) \to y$. Now we define the extended map by $\tilde{T}(x) = y$. This map is well defined because if $\{x_n\} = \{\tilde{x}_n\} \in \tilde{V_1}$ we get that

$$\|x_n - \tilde{x}_n\| \to 0 \implies \|T(x_n) - T(\tilde{x}_n)\| \to 0 \implies \tilde{T}(x) = limT(x_n) = limT(\tilde{x}_n) = \tilde{T}(\tilde{x})$$

So we now verify that it is bounded by the same constant

$$\|\tilde{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} C\|x_n\| = C\|x\|$$

Note that it is clear that it is equal on the restriction as the cannonical embedding sends elements to constant sequences. Moreover it is unique because V_1 is dense in its completion. □

Lemma 8

$C[0,1]$ is complete in the $\|\cdot\|_\infty$ topology

Proof. Take an arbitrary Cauchy sequence $\{f_n\}$. We know that we have an $\epsilon > 0$ and an N such that $n, m \geq N$ then we see that

$$\|f_n - f_m\| = \underset{x\in[0,1]}{sup)} |f_n(x) - f_m(x)| < \epsilon$$

i.e. $\{f_n(x)\} \subset \mathbb{R}$ is Cauchy for a fixed $x \in [0,1]$. So we again define a candidate function as the pointwise limit $f(x) = lim\ f_n(x)$. Note that on a compact interval continuous functions attain their maximum so we can replace supremums for straight forward maximums (in particular there will be no question if we can interchage it with a limit). For the same ϵ, N and $\forall n, m > N$ we then have

$$\underset{x\in[0,1]}{max}|f_n x - f_m x| < \epsilon \implies \underset{m\to\infty}{lim}\underset{x\in[0,1]}{max}|f_n x - f_m x| = \underset{x\in[0,1]}{max}|f_n x - fx| = \|f_n - f\|_\infty < \epsilon$$

So we need only to show that our candidate is in fact continuous. This is easy because for $x, y \in [0,1]$

$$|fx - fy| = |fx - f_n x + f_n x - f_n y + f_n y - fy| \leq |fx - f_n x| + |f_n x - f_n y| + |f_n y - fy|$$

And thus by the limit we have just computed and continuity of each f_n. In particular if we pick n sufficiently large and $x - y$ sufficiently small we can have all 3 of the summands on the right less than $\epsilon/3$. □

Definition 18

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\mathcal{F}$ a collection of functions from $X \to Y$. $\mathcal{F}$ is equicontinuous

$$\iff \forall x \in X, \forall \epsilon > 0, \exists \delta > 0 \forall f \in \mathcal{F}, d_X(x,y) < \delta \implies d_y(fx, fy) < \epsilon$$

Definition 19

$\mathcal{F}$ is uniformly equicontinuous $\iff \forall \epsilon > 0, \exists \delta > 0 \forall x, y \forall f \in \mathcal{F}\ [d(x,y) < \delta \implies \tilde{d}(fx, fy) < \epsilon]$

Let (X, d) a compact metric space and $(Y, \tilde{d})$ and arbitrary metric sapce then $\mathcal{F} \subseteq C(X, Y)$ is equicontinuous $\implies \mathcal{F}$ is uniformly equicontinuous

Definition 20

A metric space (X, d_X) is seperable $\iff \exists D \subseteq X$ where D is both countable and dense

Theorem 3

Arzelà–Ascoli Theorem:
$(X, d_X), (Y, d_Y)$ metric spaces, X seperable, $\mathcal{F}$ a family of equicontinuous functions from $X \to Y$, $\forall x \in X, \mathcal{F}(x) = \{f_n(x) | n \in \mathbb{N}\}$ is compact

$$\implies$$

For every sequence in $\mathcal{F}$ there exists a subsequence that converges pointwise to a continuous function. And $U \subseteq X$ compact $\implies$ The convergence is uniform.

There are some corollaries and other (more simple) formulations of the Arzela-Ascoli theorem.

- (f_n) a family of uniformly bounded equicontinuous functions on $[0,1] \implies$ there exists a convergent subsequence of (f_n) converging uniformly on $[0,1]$
- (X,d) is a compact metric space and $\mathcal{F} \subseteq C_b(X)$ (bounded and continuous functions from $X \to \mathbb{R}$)

$$\mathcal{F} \text{ precompact } \iff \mathcal{F} \text{ equicontinuous}$$

- (X,d) a compact metric space, $\mathcal{F} \subseteq C_b(X,E)$ (The space of continuous bounded functions from X to an arbitrary banach space E)

$$\mathcal{F} \text{ precompact } \iff \forall x \in E, \mathcal{F} \text{ is equicontinuous at } x \wedge \mathcal{F}(x) \text{ is precompact}$$

The following three lemmas were proved and key in proving AA. Let $(X,d_X),(Y,d_Y)$ metric spaces:

Lemma 9

If (Y,d_Y) is complete:
$\{f_n : X \to Y\}_{n\geq 1}$ equicontinuous and $D \subseteq X$ dense such that $\forall x \in D, \{f_n(x)\}_{n\geq 1}$ converges $\implies \forall x \in X \{f_n(x)\}_{n\geq 1}$ converges and $f(x) = lim_{n\to\infty} f_n(x)$ is continuous

Proof. Take an $x \in X$. To show that $\{f_n(x)\}$ converges it will suffice to show it is Cauchy (Y is complete). This is clear however because $\forall \epsilon > 0\, \exists \delta > 0$ such that $d_X(x,y) < \delta \implies d_Y(f_n x, f_n y) < \frac{\epsilon}{3}$ by equicontinuity and then by the denseness of $D \in X$ $\forall x \in X$ $\exists y \in D$ $|x-y| < \delta$. Thus

$$\begin{aligned} d_Y(f_n x, f_m x) &< d_Y(f_n x, f_n x) + d_Y(f_n y, f_m y) + d_Y(f_m y, f_m x) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

After choosing an appropriate $y \in D$.

It remains to show that $f = lim\ f_n$ is in fact continuous. By equicontinuity we have that $\forall \epsilon > 0\ \forall x \in X\ \exists \delta > 0\ \forall n > 0\ d_X(x,y) < \delta \implies d_Y(f_n x, f_n y) < \frac{\epsilon}{2}$. By continuity of d_Y we then have that

$$d_Y(fx, fy) = \lim_{n\to\infty} d_Y(f_n x, f_n y) \leq \epsilon/2 < \epsilon$$

□

Lemma 10

$\{f_n : X \to Y\}_{n\geq 1}$ equicontinuous and $\exists f : X \to Y$ such that $\forall x \in X$ $f_n(x) \to f(x)$. If we have in addition that X is compact then $f_n \to f$ uniformly.
i.e. For an equicontinuous family convergence pointwise on a compact set is uniform convergence.

Proof. The excersise above gives us that $\{f_n\} \cup \{f\}$ is a uniformly equicontinuous family (because X is compact). Now let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\forall n \in \mathbb{N}\ \ d_Y(f_n x, f_n y) < \frac{\epsilon}{3}\ \wedge\ d_Y(fx, fy) < \frac{\epsilon}{3}$$

Which we can do by equicontinuity. We have the following cover of X $\{B_\delta(y)\}_{y\in X}$ which using compactness we reduce to a finite subcover

$$\bigcup_{i=1}^{n} B_\delta(y_i)$$

Now using the assumed pointwise convergence on these y_i we get

$$\forall i \in [n]\ \exists N_i > 0\ \forall n > N_i\ d_Y(f_n y_i, f y_i) < \frac{\epsilon}{3}$$

Now take $N = \underset{i \in [n]}{max} N_i$ and we get that

$$\begin{aligned} |f_n x - fx| &\leq |f_n x - f_n y_i| + |f_n y_i - f y_i| + |f y_i - fx| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

where the y_i is selected such that $x \in B_\delta(y_i)$ and then we bound each of the terms using (equi)continuity and convergence pointwise. The final step is that for $n > N$

$$\|f_n - f\|_\infty = \underset{x \in X}{sup}\ d_Y(f_n x, fx) = \underset{x \in X}{max}\ d_Y(f_n x, fx) < \epsilon$$

Because it can be made less than ϵ for all $x \in X$ and therefore the max must be less as well (note that a sup may have changed it to $\leq$ which would still be fine, however we have a genuine max here by compactness of X). □

Lemma 11

D a countable set and $\mathcal{F} = \{f_n : D \to Y\}$. If $\forall x \in D, \overline{\{f_n(x) : n \in \mathbb{N}\}}$ is compact then $\exists$ a subsequence of $\mathcal{F}$ convergent on all of D.

Proof. First list the elements of $D = \{x_1, x_2, ...\}$. Next we use the precompactness of $\{f_n(x) : n \in \mathbb{N}\}$ for every $x \in D$ to get that $\{f_i(x_1)\}_{i \geq 1}$ has a convergent subsequence. We label it as $\{f_{1_k} x\}_{k \geq 1}$. we apply precompactness again to see that $\{f_{1_k}(x_2)\}_{k \geq 1}$ admits a convergent subsequence, call it $\{f_{2_k} x_2\}_{k \geq 1}$. We proceed like this to see that $\{f_{n_n}\}_{n \geq 1}$ converges on all of D. □

2 Hilbert Spaces

Definition 21

A vector space V over the field $\mathbb{C}$ with a bilinear form $(,) : V \times V \to \mathbb{C}$ is an inner product space (with inner product $(,)$) if the bilinear form satisfies: $\forall v, u, w \in V, \forall \alpha \in \mathbb{C}$

- $(v, v) \geq 0$
- $(v, v) = 0 \iff v = 0$
- $(v, u + w) = (v, u) + (v, w)$
- $(v, \alpha u) = \alpha(v, u)$
- $(v, u) = \overline{(u, v)}$

Lemma 12

An inner product on V induces a norm on V given by $\|v\| = \sqrt{(v, v)}$

Definition 22

If the inner product space $(V, (-, -))$ is complete with respect to the metric induced by the norm induced by the inner product, then we call it a Hilbert space

Definition 23

$(V, (-, -))$ an inner product space, $x, y \in V$

$$x \text{ orthogonal to } y \iff x \perp y \iff (x, y) = 0$$

Definition 24

A collection of elements, $\{x_\alpha\}_{\alpha \in A}$, from a Hilbert space $\mathcal{H}$ are orthonormal $\iff$ $(x_a, x_b) = I(a = b)$

Lemma 13

$x \in V$ an innerproduct space and $\{x_i\}_{1 \leq i \leq N} \subseteq V$ an orthonormal set $\implies$ $\forall x \in V, \|x\|^2 = \sum_{i=1}^{N} |(x_i, x)|^2 + \|(x - \sum_{i=1}^{N} (x_i, x)x_i)\|^2$

Theorem 4

Bessels Inequality: Under the same assumptions as above $\|x\|^2 \geq \sum_{i \in I} |(x_i, x)|^2$

Theorem 5

Cauchy Schwartz Inequality: For an inner product space V and any $x, y \in V$

$$|(x, y)| \leq \|x\| \cdot \|y\|$$

Proof. Case 1 is when $y = 0$ which gives $0 \leq 0$ immediately. So we may now assume $y \neq 0$. Then the singleton set $\{\frac{y}{\|y\|}\}$ is trivially orthonormal and so by Bessels inequality

$$\|x\|^2 \geq |(\frac{y}{\|y\|}, x)|^2 \implies \|x\|^2\|y\|^2 \geq |((y, x))|$$

By linearity of the innerproduct. □

The key example of a Hilbert space is $L^2[a,b]$ we will define this space as the metric completion of continuous functions with a given metric. Specifically

$$L^2[a,b] = completion(cts([a,b],\mathbb{C}), d(f,g) = \sqrt{\int_a^b |f-g|^2})$$

Completion Recall that for an arbitrary metric space (X,d) we can construct its completion $\tilde{X} = \{\text{Cauchy sequences in X}\}/\sim$ where $\sim$ is the equivilence relation identifying $(x_i) \sim (y_i) \iff d(x_i,y_i) \to 0$ for Cauchy sequences. This is also a metric space with the metric $\tilde{d}(x,y) = lim_{i\to\infty} d(x_i,y_i)$

We have the natural inclusion $\iota : X \hookrightarrow \tilde{X}$ such that $\iota(X)$ is dense in $\tilde{X}$ and $d(x,y) = \tilde{d}(\iota x, \iota y)$

Definition 25

An isomorphism of Hilbert spaces is an innerproduct preserving bijection.

Definition 26

For a Hilbert space $\mathcal{H}$ we define its dual as $\mathcal{H}^* = \{\ell : \mathcal{H} \to \mathbb{C} | \ell \text{ is linear and bounded }\}$

Lemma 14

Let $(\mathcal{H},(-,-))$ a Hilbert space and $m \subseteq \mathcal{H}$ a closed subspace and $x \in \mathcal{H}$ arbitrary $\implies$

- $\exists z \in m, \|z-x\| = inf\{\|y-x\| : y \in m\}$
- This z is unique
- $x - z \perp m$ i.e. $\forall w \in m, (x-z,w) = 0$
  i.e. $\mathcal{H} = m \oplus m^\perp$

Proof. Take an arbitrary $x \in \mathcal{H}$, then let $d = \underset{y\in m}{inf} \|x-y\|$. Now take a sequence in M $\{y_i\}_{i\geq 1}$ such that $lim_{n\to\infty}\|x-y_n\| = d$. If we can show this sequence is Cauchy we will have shown it converges in M and therefore there will be an element in m that attains the infimum required. So examine

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|y_n - x - (y_m - x)\|^2 \\
{\color{red}\star} &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|2x - y_n - y_m\|^2 \\
&\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - (\|x-y_n\| + \|x - y_m\|)^2 \\
\implies \lim_{n,m\to\infty} \|y_n - y_m\| &\leq \lim_{n,m\to\infty} 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - (\|x-y_n\| + \|x - y_m\|)^2 \\
&\leq 2d^2 + 2d^2 - 4d^2 = 0
\end{aligned}$$

Note that $\star$ is an application of the parrallelogram law on NLS

$$2\|a\|^2 + 2\|b\|^2 = \|a-b\|^2 + \|a+b\|^2$$

As stated earlier, we have now shown that this sequence is Cauchy in m and therefore converges in m (m is closed in a Hilbert space). Thus we have existence. For uniqueness assume that both $y, y' \in m$ attain the infimum

$$\|y - y'\| \leq 2\|y-x\|^2 + 2\|y'-x\|^2 - (\|x-y\| + \|x-y'\|)^2 = 0$$

Using the same expansion as above.

Finally we show that $x - z \perp m$. To do this begin by taking an arbitrary $y \in m, \alpha \in \mathbb{C}$. A simple calculation shows

$$d^2 = inf_n\|x-n\|^2 \leq \|x-(z+\alpha y)\|^2 = \|(x-z)-\alpha y\|^2 = \|x-z\|^2 - \alpha(x-z,y) - \bar{\alpha}(y,x-z) + |\alpha|^2\|y\|^2$$

Note that m is a vector subspace allowing us to take linear combinations of $z, y \in m$ without leaving m.

We recognise that $\|x-z\|^2 = d^2$ and so by subtract it from both sides of the inequality to get

$$2Re(\alpha(x-z,y)) \leq |\alpha|^2\|y\|^2$$

We now deal with two cases seperately. Suppose $\alpha \in \mathbb{R}$:

$$\begin{aligned} &2\alpha R(x-z,y) \leq \alpha^2\|y\|^2 \\ \iff\ &2Re(x-z,y) \leq \alpha\|y\|^2 \end{aligned}$$

Valid for any α. Namely we can make $Re(x-z,y) < \epsilon$ for any ϵ and therefore it must be zero.

Next consider $t \in \mathbb{R},\ \alpha = ti$

$$\begin{aligned} &2\alpha R(x-z,y) \leq \alpha^2\|y\|^2 \\ \iff\ &2Im(x-z,y) \leq \alpha\|y\|^2 \end{aligned}$$

And again we see that the imaginary part of $x - z$ goes to zero. So we get that

$$\forall y \in m\ (x-z,y) = 0$$

i.e. $x - z \perp m$ □

Theorem 6

Riesz Representation Theorem: There is an isomorphism, $\phi : \mathcal{H} \to \mathcal{H}^*$, between every hilbert space and its dual. Moreover this isomorphism is given by $v \mapsto (v, -)$

We will denote $\phi(v)(x) = \ell_v(x) = (v,x)$

Proof. We will show that we have an isometry and a bijection.

Why dont we prove that it preserves the inner product. In fact what is the inner product on the dual of $\mathcal{H}$

First if $x = 0$ we have immediately that $\ell_x = 0$ and so trivially $\|\ell_x\| = \|x\| = 0$.

Next for $x \neq 0$ we have that

$$\begin{aligned} \|\ell_x\| &= \sup_{\|y\|=1} \|\ell_x(y)\| \\ &= \sup_{\|y\|=1} \|(x,y)\| \\ &\leq \sup_{\|y\|=1} \|x\|\|y\| \\ &\leq \|x\| \end{aligned}$$

$$\begin{aligned} \|\ell_x\| &= \sup_{\|y\|=1} \|\ell_x(y)\| \\ &\geq |\ell_x(\frac{x}{\|x\|})| \\ &= \frac{\|x\|^2}{\|x\|} \\ &= \|x\| \end{aligned}$$

From it being an isometry we get that $\Phi(x) = 0 \iff x = 0$ i.e. Φ is injective.

For surjectivity let $T \in \mathcal{H}^*$ and $N = ker(T)$. Recall that N is always closed (Hilbert spaces are metric spaces and therefore Hausdorff and so points are closed, and the preimage of closed sets under continuous maps is closed). We have established that if $T = 0$ then $\Phi(0) = T$ so we may assume that $T \neq 0$ i.e. $N \neq \mathcal{H}$. We now employ the earlier result that $\mathcal{H} = N \oplus N^\perp$.

So let $x_0 \in N^\perp \setminus \{0\}$. I will construct an element such that T is pairing with that element.

First I claim that $N^\perp = span_{\mathbb{C}}\{x_0\}$, this can be seen by letting $y \in \mathcal{H}$ then

$$T(y) = \frac{Ty}{Tx_0} Tx_0 = T(\frac{Ty}{Tx_0} x_0)$$

But we also have that $y = y - \frac{Ty}{Tx_0}x_0 + \frac{Ty}{Tx_0}x_0$ so we can conclude that $y - \frac{Ty}{Tx_0}x_0 \in N$ and $\frac{Ty}{Tx_0}x_0 \in N^\perp$. So for arbitrary $y = n + n^\perp$ we have that $n^\perp = \frac{Ty}{Tx_0}x_0$ and so we have the claim.

Now $\forall y \in \mathcal{H}\ y = y'\alpha x_0$ and

$$T(y) = \alpha T(x_0) = \alpha T(x_0)\frac{(x_0, x_0)}{\|x_0\|^2} = (\frac{\bar{T}(x_0)}{\|x_0\|^2}x_0, \alpha x_0)$$

Note that we only need to show that it is pairing with an element of the perp because if we expand the full pairing we get it is zero anyway. □

It follows from this that any sesquilinear (think bilinear but conjugate constants from the first slot) form B from a Hilbert space to $\mathbb{C}$ that also satisfies the bound $|B(x, y)| \leq C\|x\|\|y\|$ will be given by pairing, in particular $\exists A : \mathcal{H} \to \mathcal{H}$ a BLT such that $B(x, y) = (Ax, y)$

Proof. □

Lemma 15

The pairing map above, ℓ_v is a bounded and linear map

Definition 27

A subset $S \subseteq \mathcal{H}$ of an inner product space is a maximal orthonormal set $\iff$

- S is an orthonormal set
- $\forall S' \subseteq \mathcal{H}$ such that S' is an orthonormal set $[S \subseteq S' \implies S = S']$

Definition 28

An orthonormal basis of a Hilbert sapce is a maximal orthonormal subset (ONB)

Lemma 16

$\{x_\alpha\}_{\alpha \in A} = S \subseteq \mathcal{H}$ an ONB of a Hilbert space $\implies$

- $\forall x \in \mathcal{H}, x = \sum_{\alpha \in A}(x_\alpha, x)x_\alpha$ such that only countably many of the (x_α, x) are non-zero
- $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{\alpha \in A} |(x_\alpha, x)|^2$

Proof. It is clear that all of the following sets are finite

$$B_n = \{\alpha \in A : |(x_\alpha, x)| > \frac{1}{n}\}$$

(If any were infinite then by Bessells inequality $\|x\|^2 > \infty$). But then we have that

$$x = \sum_{\alpha \in \cup_{n\geq 1} B_n} (x_\alpha, x)x$$

Which is a countable sum (countable union of finite sets is countable), and so the other (x_α, x) terms must be zero (again appealing to Bessells inequality to argue that they must be precisely zero). □

Definition 29

A Hilbert space is called seperable if there exists a countable orthonormal basis.

Lemma 17

$\mathcal{H}$ a Hilbert space

$$\mathcal{H} \text{ is seperable as a Hilbert space} \iff \mathcal{H} \text{ is seperable as a metric space}$$

Proof. For a seperable Hilbert space we take the union of all the $\mathbb{Q}$ spans of finite collections of elements of the orthonormal basis. This is countable and dense.

The other direction uses Zorns lemma and ordering sets of linearly independent vectors. □

Theorem 7

Any two seperable Hilbert spaces are isomorphic

2.1 Banach Spaces

Definition 30

An isometry of a NLS is a bijective norm preserving map.

Definition 31

We can define for two NLS $(V_1, \|-\|_1), (V_2, \|-\|_2)$

$$\mathcal{L}(V_1, V_2) = \{T : V_1 \to V_2 | T \text{ is bounded and linear } \}$$

This is also a normed linear space with the induced norm

$$\|T\|_{\mathcal{L}(V_1,V_2)} = inf\{C > 0 : \forall v \in V_1, \|Tv\|_2 \leq C\|v\|_1\} = sup\{\|Tv\|_2 : \|v\|_1 = 1\}$$

Lemma 18

V_2 complete $\implies \mathcal{L}(V_1, V_2)$ is complete

Proof. Start with a Cauchy sequence $\{T_n\}_{n\geq 1} \subseteq \mathcal{L}(V_1, V_2)$. For any $x \in V_1$ we have that $T_n(x)$ is Cauchy (immediate from boundedness of the T_n). So we can define a candidate function pointwise as

$$T(x) = \lim_{n\to\infty} T_n(x)$$

It is clear that this candidate is linear and boundedness is shown by

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \|T_n\|\|x\| \leq C\|x\|$$

Where we have used that Cauchy sequences are bounded on NLS's. So our candidate function is in fact a BLT. We need only to show that the sequence converges in norm now.

$$\|(T_n - T)x\| = \lim_{m\to\infty} \|(T_n - T_m)x\| \leq \lim_{m\to\infty} \|T_n - T_m\|\|x\| < \epsilon\|x\|$$

Where we get the final bound for a sufficiently large $n > N$ and the Cauchyness of $\{T_n\}$. So we have shown that $\|T - T_n\| \to 0$ and we are done. $\square$

Theorem 8

Any finite dimensional vector spaces of the same dimension are isomorphic

An interesting set that we can consider is $\ell^p = \{(x_i)_{i>0} | x_i \in \mathbb{C}, \sum_{i>0} |x_i|^p < \infty\}$ i.e. complex sequences that are p^{th} power summable.

All such ℓ^p spaces are Banach spaces with the norm given by $\|x\|_p = (\sum_{i>0} |x_i|^p)^{\frac{1}{p}}$

Lemma 19

ℓ^2 with the following inner product $(x, y) = \sum_{i>0} \bar{x}_i y_i$ is a Hilbert space. Moreover it is seperable

Because ℓ^2 is a seperable Hilbert space there must be an isomorphism between it and all other seperable Hilbert spaces. In fact this map is given by $v \mapsto \{(x_{\alpha_i}, v)\}_{i\geq 1}$ where $(x_{\alpha_i})_{i\geq 1}$ is a countable subsequence of a given orthonomal basis on $\mathcal{H}$ such that $v = \sum_{i\geq 1}(x_{\alpha_i}, v)v$ (which exists because $\mathcal{H}$ is seperable). This is clearly a well defined complex sequence, moreover $\|v\|^2 = \sum |(x_{\alpha_i}, v)|^2 < \infty$ which implies it is in ℓ^2. We can also check that this is a bijection and that it preserves the innerproduct.

Another example is $L^p(\mu)$, the space of p^{th} integrable functions on the measure space μ. We can give these the norm of p^{th} power integration and rooting i.e.

$$\|f\|_p^p = \int |fx|^p dx$$

On L^p space we have for $p, q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\text{Holder: } \int \bar{f}g \leq \|f\|_{L^p}\|g\|_{L^q}$$

$$\text{Minkowski: } \|f + g\| \leq \|f\| + \|g\|$$

$L^2(\mathbb{S}^1)$, the space of square integrable periodic functions, has the mapping into $\ell^2_{\mathbb{Z}}$ giving the Fourier coefficients.

In particular $\{e^{i\theta n}\}_n$ forms an ONB of $L^2(\mathbb{S}^1)$ and so after pairing with a function we get

$$f(\theta) = \sum_n (e^{i\theta n}, f)e^{i\theta n} = \sum_n e^{i\theta n} \int e^{i\theta n} f(\theta) d\theta$$

So each function is identified by the sequence of coefficients here.

Two more examples are $\ell^\infty = \{$ bounded complex sequences $\}$ and $c_0 = \{(x_i)_{i>0} : lim_{i\to\infty} x_i = 0\}$

Lemma 20

ℓ^∞ and c_0 are Banach spaces when given the sup norm

Lemma 21

$$(c_0)^* \cong \ell^1$$
$$(\ell^1)^* \cong \ell^\infty$$

Proof. given on page 46 as an example. Also revise assignment □

Theorem 9

$\frac{1}{p} + \frac{1}{q} = 1 \implies$

- $(\ell^p)^* \cong \ell^q$
- $((\ell^p)^*)^* \cong \ell^p$

Moreover the Holder inequality still applies

Explicitly the isometry sending $\ell_p \to (\ell_q)^*$ is given by $x \mapsto L_x$ where $L_x : \ell_p \to \mathbb{C}, y \mapsto \sum \bar{x}_i y_i$ (as expected)

Proof. That this is an isometry and the map of pairing with something in lp gives something in lq

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□

So in summary we should have the following spaces at our disposal

- $\ell_\infty = \{a : sup_n |a_n| < \infty\}$ (bounded sequences)
- $c_0 = \{a : lim_{n\to\infty} a_n = 0\}$ (sequences going to zero)
- $\ell_p = \{a : (\sum |a_n|^p)^{\frac{1}{p}} < \infty\}$ (p^{th} summable sequences)
- $s_p = \{a : lim_{n\to\infty} n^p a_n = 0\}$
- $f = \{a : a_n = 0$ for all but finitely many n $\}$

We have the following series of inclusions for these spaces

$$f \subset s_p \subset \ell_q \subset c_0 \subset \ell_\infty$$

ℓ_q, c_0, ℓ_∞ can be made into Banach spaces as outlined above. s is a Frechet space (to be discussed later) and f is dense in all except ℓ_∞

Definition 32

For a NLS $(X^*)^* = \mathcal{L}(X^*, \mathbb{C})$

There is a canonical map $\phi : X \to (X^*)^*$ given by $x \mapsto \tilde{x}$ where $\tilde{x} : X^* \to \mathbb{C}$ sending $\ell \mapsto \ell(x)$ i.e. $\phi(x)(\ell) = \ell(x)$

Theorem 10

If X above is a Banach space then

- ϕ is injective
- $\|\phi(x)\|_{(X^*)^*} = \|x\|$

Note that when this map is surjective i.e. an isomorphism we call the Banach space reflexive. We showed above that $(c_0)^{**} = (\ell_1)^* = \ell_\infty \neq c_0$ and thus c_0 is not a reflexive Banach space

Proof. This proof relies on Hahn Banach theorem, proved below. □

Definition 33

A function from a vector space to $\mathbb{R}$, $p : V \to \mathbb{C}$ is convex $\iff \forall y, y' \in V \forall \alpha \in [0,1], p(\alpha y + (1-\alpha)y') \leq \alpha p(y) + (1-\alpha)p(y')$

Theorem 11

Hahn-Banach Theorem: $p : V \to \mathbb{R}$ a convex function, $W \subseteq V$ and $\lambda : W \to \mathbb{R}$ linear such that $\forall w \in W, \lambda(w) \leq p(w)$

$$\implies \exists \Lambda : V \to \mathbb{R}, \Lambda|_W = \lambda \wedge \forall v \in V, \Lambda(v) \leq p(v)$$

Proof. The idea is to extend one vector at a time (and then apply Zorns lemma; that a partially ordered set where every chain has an upper bound has at least one maximal element).

Only partially proven in lecture/notes TUESDAY MARCH 22. Highly doubt it will be on the exam

□

There are several convenient corroleries

Lemma 22

X a NLS and $Y \subseteq X, \lambda \in Y^* \implies \exists \Lambda \in X^*$ such that $\Lambda|_Y = \lambda$ and $\|\Lambda\|_{X^*} = \|\lambda\|_{Y^*}$

Proof. Let $p(x) = \|\lambda\|_{Y^*}\|x\|$ and note that this is a convex function. □

Lemma 23

$y \in X \implies \exists \Lambda \in X^*, \Lambda(y) = \|\Lambda\|_{X^*}\|y\|$

Proof. Take $Y = span(y)$ and define a function on this linear subspace by $\lambda(ay) = a\|y\|$. Note that $\|\lambda\| = \sup_{\|ay\|=1} \|ay\| = 1$ and so applying the lemma above we get that $\exists\Lambda$ an extension with $\|\Lambda\| = \|\lambda\| = 1$. Moreover because $y \in Y$ we have that

$$\Lambda(y) = \lambda(y) = 1 \cdot \|y\| = \|\Lambda\|\|y\|$$

□

2.2 Banach Spaces as a Category with Bounded Linear Transformations

We would like to be able to think about Banach spaces as a category, with the set of morphism being the bounded linear transformations. To do this we will need to set up some theory to prove that bounded linear transformations meet the requirements for being the morphism of this category. Namely when we compose bounded linear transformations (BLT's) we get a BLT again and the inverse of a bijective BLT is again a BLT.

Definition 34

A set $A \subseteq X$ for (X,d) a metric space, is nowhere dense (or meagre) $\iff \bar{A}$ has an empty interior. i.e.

$$\bigcup_{K \subseteq \bar{A}, K \text{ open}} K = \emptyset$$

Theorem 12

Baire Category Theorem:
(X, d) a complete metric space such that $X = \cup_{n \in \mathbb{N}} A_n \implies \exists n, A_n$ is not nowhere dense
Equivilently: $\forall n, A_n$ are nowhere dense $\implies X \neq \cup_{n \in \mathbb{N}} A_n$

Proof. Assume that A_n are nowhere dense for all n. Now because A_1^C has a dense interior $\exists x \in A_1^C$ $\exists r > 0$ $B_r(x) \subset A_1^C$ which then implies that we can take a ball $B_1 = B_{min(\frac{r}{2},1)}(x)$ such that $\bar{B}_1 \cap A_1 = \emptyset$. We carry on this construction taking B_n to be an open ball such that $\bar{B}_n \cap A_n = \emptyset$, $B_n \subseteq B_{n-1}$ and $radius(B_n) < \frac{1}{2^{n-1}}$.

At each stage we take an element to get a sequence $\{x_n\}_{n \geq 1}$, where $x_n \in B_n$. Claim that this sequence is Cauchy (so converges by completeness) and that the limit does not lie in the union of the A_n's. This would prove that $X \neq \cup_{n \in \mathbb{N}} A_n$.

Proof of this claim: Because $\forall i > j$ $x_i \in B_j$, we can take an N sufficiently large such that $\forall n > m > N$

$$d(x_n, x_m) < \frac{2}{2^{n-1}} < \epsilon$$

and so Cauchyness is clear.

Using the same property we have that for $m > n$ $x_m \in \bar{B}_n$ and therefore $lim_{n \to \infty} x_n \in \bar{B}_n$ (closed balls contain their limits (theyre closed)). But by construction $\bar{B}_n \cap A_n = \emptyset$ for every n. So $x \notin \cup A_n$.
□

Theorem 13

Uniform Boundedness Principle: X, Y Banach spaces and $\mathcal{T} \subseteq \mathcal{L}(X, Y), \forall x \in X, \{\|T(x)\| : T \in \mathcal{T}\}$ is bounded

$$\implies \exists C > 0 \forall T \in \mathcal{T}, \|T\| < C$$

Proof. First consider the sets $A_n = \{x : \|Tx\| < n \; \forall T \in \mathcal{T}\}$, by assumption we have that $X = \cup A_n$ (because $\forall x \in X$ the image is bounded).

By Baire category theorem then we have that $\exists A_n$ with a nonempty interior, i.e. we have $\exists x_0 \in A_n \exists r > 0$ $B_r(x_0) \subseteq A_n$.

Now consider an $x \in B_r(0)$. We can write this as $x = x + x_0 - x_0$ thus

$$\begin{aligned} \|Tx\| &\leq \|T(x + x_0) - T(x_0)\| \\ &\leq \|T(x + x_0)\| + \|T(x_0)\| \\ &\leq n + n \\ &\leq n \end{aligned}$$

Without loss of generality we can assume that $\|x\| = 1$ (by scalling by the norm) then we see that $\frac{x\epsilon}{2} \in B_\epsilon(0)$ which gives

$$\|T(\frac{x\epsilon}{2})\| \leq 2n$$
$$\implies \|T(x)\| \leq \frac{4n}{\epsilon}$$
$$\implies \|T\| \leq \frac{4n}{\epsilon}$$

□

Theorem 14

Open Mapping Theorem: A BLT between Banach spaces $T : X \to Y$ surjective $\implies$ T is an open mapping
(Sends open sets to open sets)

Proof. □

Theorem 15

Inverse Mapping Theorem: A BLT between Banach spaces $T : X \to Y$ is bijective $\implies$ T^{-1} is continuous

Proof. □

Theorem 16

$T : X \to Y$ a linear map between Banach spaces

$$T \text{ bounded} \iff \{(x, T(x))|x \in X\} \subseteq X \times Y \text{ is closed}$$

Proof. Assume the graph is closed, then by linearity we get that it is a closed subspace of a Banach space $X \times Y$ and therefore itself a Banach space under the norm

$$\|(x, Tx)\| = \|x\| + \|Tx\|$$

Consider the two projection maps π_1, π_2. These are both continuous and p_1 is a bijection. Thus by the inverse mapping theorem we have that π_1^{-1} is continous. But

$$T = \pi_2 \circ \pi_1^{-1}$$

So we are done. For the converse we can use that the image of a continuous function is always closed because

$$lim\ f(x_n) = f(lim\ x_n)$$

□

Definition 35

The direct sum of an arbitrary collection of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ is

$$\bigoplus_{\alpha \in A} \mathcal{H}_\alpha = \{\{x_\alpha | x_\alpha \in \mathcal{H}_\alpha\}_{\alpha \in A} | \sum_{\alpha \in A} \|x_\alpha\| < \infty\}$$

2.3 Quotients of Banach Spaces

Theorem 17

A NLS is complete $\iff$ every absolutely summable sequence is summable

Proof. Assume X is Banach and a sequence $\{x_n\}$ is absolutely summable. Then we have that the partial sums form a Cauchy sequence because

$$\|S_N - S_M\| = \|\sum_{n=N}^{M} x_n\| \leq \sum_{n=N}^{M} \|x_n\| \leq \sum_{n \geq N} \|x_n\| \to 0$$

And thus S_N converges (ie the sequence is summable)

Reverse implication

□

Definition 36

Recall for finite dimensional vector spaces, $V^m \subset W^n, m < n$, we may take their quotient and get a vector space $W/V = \{w + V | w \in W\} = \{\{v + w | v \in V\} | w \in W\}$ with the vector space operations defined by

i) $[w] + [w'] = [w + w']$ ii) $\lambda[w] = [\lambda w]$

where $[w], [w'] \in W/V$ and λ is an element of the appropriate field.

Theorem 18

For an arbitrary Banach space X and a closed linear subspace $M \subseteq X$ we can define the quotient space, in the same way. Moreover we can give this quotient space a norm

$$\|[x]\|_{X/M} = inf_{m \in M} \|x - m\|_X$$

which in tern makes the quotient space X/M a Banach space (complete).

Proof.

Show that this is in fact a norm

We will use the previous lemma, so take an absolutely summable sequence $\{[x_i]\}_{i \geq 1}$ i.e.

$$\sum_{i \geq 1} \|[x_i]\| = \sum_{i \geq 1} \underset{m \in M}{inf} \|x_i - m\| < \infty$$

For each n we select an element of the closed subspace m_n such that $\|x_n - m_n\| \leq 2 \underset{m \in M}{inf} \|x_n - m\|$ (one clearly exists or the inf would be bigger). Thus

$$\sum_{n=1}^{N} \|x_n - m_n\| \leq 2 \sum_{n=1}^{N} \|[x_n]\|$$

Which by assumption is finite in the limit. Thus we have that $x_n - m_n$ is absolutely summable, we now invoke the above lemma on X which is a Banach space to imply that $x_n - m_n$ is summable.

Now define $y = \sum_{i \geq 1} x_n - m_n$. Claim that $[y] = \sum_{i \geq 1} [x_i]$ which would show that this is summable and therefore the quotient space was Banach.

We can prove this claim by

$$\|[y] - \sum_{i=1}^{N} [x_i]\| = \|[y] - \sum_{i=1}^{N} [x_n - x_m]\| = \|[y - \sum_{i=1}^{N} x_n - x_m]\| \xrightarrow{N \to \infty} 0$$

□

Definition 37

A family of functions $\mathcal{F} = \{f_n\} \subseteq C_{\mathbb{R}}(X)$ on a space X seperates points iff $\forall x, y \in X, \exists f \in \mathcal{F}, f(x) \neq f(y)$

Jesse Mentioned:
Stone-Weirstrass
Urysohs Lemma
Tzitze Extension Theorem

3 Weak Topologies

Motivation: We have in ℓ^2 an infinite dimensional orthonormal basis given by $e_i = (0, ..., 0, 1, 0, ...)$ where the 1 is in the i^{th} position.

$lim_{i\to\infty} e_i$ Does not exist in the metric topology on ℓ^2 since $\|e_i - e_j\|^2 = 2$.

If we look at the dual however there is a sense in which it converges, namely for $\lambda \in (\ell^2)^* = \ell_x$ for some $x =\in \ell^2$ we have that $\lambda(e_i) = (x, e_i) = \sum_{i\geq 1} \bar{x}_i e_i = \bar{x}_i \to 0$. This is clear because $x \in \ell^2$ and therefore $\sum |x_i|^2 < \infty$ and so the terms must go to zero, $|x_i|^2 \to 0$

We will say here that the $\lambda(e_i)$ converge weakly to 0, moreover the translates $\lambda(y + e_i)$ converge weakly to y.

Definition 38

For a set S, a topological space $(\mathcal{T}, \mathcal{U})$ and some collection $\mathcal{F} \subset Func(S, \mathcal{T})$ we define the $\mathcal{F}$-weak topology on S to be the weakest topology on S such that $\forall f \in \mathcal{F}, f$ is continuous.

What do the open sets of the weak topology look like? Let $\mathcal{V}$ be the $\mathcal{F}$ weak topology. Given an $\mathcal{O} \in \mathcal{U}$ and an $f \in \mathcal{F}$ then $f^{-1}(\mathcal{O}) \in \mathcal{V}$ by definition. We also know that $\mathcal{V}$ is generated by arbitrary unions of sets of the form $f_1^{-1}(\mathcal{O}_1) \cap ... \cap f_n^{-1}(\mathcal{O}_n)$ where $n \in \mathbb{N}, f_1, ..., f_n \in \mathcal{F}, O_1, ..., O_n \in \mathcal{U}$. i.e generated by arbitrary unions over finite intersections of preimages of opens under maps from $\mathcal{F}$.

Definition 39

The weak topology on a Banach space X is the $\mathcal{F}$-weak topology given to X when $\mathcal{F} = X^*$

Given some $\lambda_1, ..., \lambda_n \in X^*$ we can define $N_{\lambda_1,...,\lambda_n,\epsilon} = \lambda_1^{-1}(B_\epsilon(0)) \cap ... \cap \lambda_n^{-1}(B_\epsilon(0)) = \{x \in X || \lambda_i(x)| < \epsilon\}$. These $N_{\lambda_1,...,\lambda_n,\epsilon}$ for a neighbourhood basis of 0 (varying over all $\lambda \in X^*$ and ϵ).

Theorem 19

For the weak topology on a Banach space X there is a neighbourhood basis given by $N_{\lambda_1,...,\lambda_n,\epsilon,x_0} = \{x \in X || \lambda_i(x - x_0)| < \epsilon, i = 1, ..., n\}$

Theorem 20

The Weak topology on $\mathbb{R}^n$ is the metric topology

Theorem 21

The weak topology is not metrisable on ℓ^2 (ie no metric exists such that the weak topology is the topology induced by that metric)

Theorem 22

The weak topology on a Banach space is weaker than the norm topology.

Proof. We have proved that the BLT's in X^* are continuous in the norm topology and the weak topology is the weakest such that this is the case. $\square$

Theorem 23

Any weakly convergent sequence is norm bounded in a Banach space

Proof. First a sequence converging in the weak topology $x_n \xrightarrow{w} x$ means that $\forall \mathcal{U}$ weakly open set such that $x \in \mathcal{U}$ we have that $\exists N > 0 \; \forall n > N \; x_n \in \mathcal{U}$.
We take such an arbitrary weakly convergent sequence that we assume for simplicity converges to 0, i.e. $x_n \xrightarrow{w} 0$ (we can translate the argument to any point with only slightly more effort).

The weak convergence tells us that $\forall \ell \in X^* \; \exists N > 0 \; n > N \implies x_n \in N_{\ell,\epsilon}$ i.e. $|\ell(x_n)| < \epsilon$

Recall the isometric embedding of $X \hookrightarrow (X^*)^*$. This allows us to think of the sequence as a sequence in the double dual of X.

$$\implies |\ell(x_n)| = |\tilde{x}_n(\ell)| < \epsilon$$

Thus we have that for any $\ell \in X^*$ the sequence $\{|\tilde{x}_n(\ell)|\}$ is bounded.
Applying the uniform boundedness principle we get that

$$\|x_n\|_X = \|\tilde{x}_n\|_{X^{**}} < C$$

Where C is an n independent constant. $\square$

Theorem 24

X is Hausdorff in the weak topology

Is this for X Banach or a general NLS. Why why not? So for finite dim the weak top on a hilbert space is the metric topology, for seperable the weak topology is not metrisable, what about nonseperable? what about a similar case discussion for Banach spaces?
So apparently for Banach spaces too all the finite dim ones are just Rn? But not so for infinite dim.
What is the case for metric spaces? I think that they are not isomorphic even in the finite dim case.
Is there an isometry if the metrics are equivilent

Proof.

exercise using Hahn Banach

$\square$

In fact if we give X the weak topology from $Y \subseteq X^*$ and Y seperates points then X is Hausdorff in the weak topology.

Theorem 25

For a hilbert space $\mathcal{H}$ and an ONB $\{x_\alpha\}_{\alpha \in A}$

$$\{y_n\}_{n\geq 1} \subseteq \mathcal{H} \text{ converges weakly to } y \iff \forall n \|y_n\| < C \wedge \forall \alpha \in A, (x_\alpha, y_n) \xrightarrow{\|\cdot\|} (x_\alpha, y)$$

ie a sequence converges weakly if and only if it is uniformly bounded and converges in the norm

Proof. $\square$

4 Locally Convex Spaces

Definition 40

$C^k[a,b]$ the space of k times differentiable functions with the norm given by $\|f\|_{C^k} = \sum_{j=0}^{k} sup_{x\in[a,b]}|\frac{\partial^j f}{\partial x^j}(x)|$

This $C^k[a,b]$ is in fact Banach with this norm.

Definition 41

The space of smooth functions is $C^\infty[a,b] = \cap_{k\geq 1} C^k[a,b]$

We are going to topologies this space of smooth functions but clearly the norm on on any of the C^k spaces is inadequate because it doesnt hold any information about the higher derivatives. To do this we will use the theory of weak topologies, seminorms and locally convex spaces.

Definition 42

A topological vector space is a vector space with a topology such that the two vector space operations are continuous and points are closed.

Note that the closed points condition will imply that any TVS is Hausdorff (although it is not itself suffiecient). This is the condition of being T_1

Definition 43

A seminorm on a vector space $\rho : V \to \mathbb{R}$ is a function satisfying all the norm axioms except potentially positivity i.e. $\rho(x) = 0 \not\Leftrightarrow x = 0$

- -

A seminorm on a space V is a map $\rho : V \to [0,\infty)$ such that $\forall x, y \in V, \forall \lambda \in \mathbb{C}$

- $\rho(\lambda x) = |\lambda|\rho(x)$
- $\rho(x+y) \leq \rho(x) + \rho(y)$

Note that norms are seminorms but the reverse is not necessarily true.

Definition 44

A family of seminorms $\{\rho_\alpha : V \to \mathbb{R}\}_{\alpha\in A}$ seperates points $\iff \forall v \in V, \exists \alpha \in A, \rho_\alpha(v) > 0$

Alternitively

$$\forall \alpha \in A, \rho_\alpha(x) = 0 \implies x = 0$$

Definition 45

The locally convex topology on a vector space V given a family of seminorms that seperate points is the $\mathcal{F}$ weak topology given by taking the set of functions $\mathcal{F} = \{x \mapsto \rho_\alpha(x - x_0)\}_{\alpha\in A, x_0\in V}$

For a point in such a locally convex vector space x_0 we have a neighbourhood basis given by

$$N_{\alpha_1,\ldots,\alpha_n,\epsilon,x_0} = \{x \in V : \rho_{\alpha_i}(x - x_0) < \epsilon, \forall i \in [n]\}$$

To relate this to the weak topology section we can see that for a vector space X and a $Y \subseteq X*$ that seperates points the weak topology given to X by Y is the same as the locally convex topology given to X by the family of seminorms $\{\rho_\ell : \rho_\ell(x) = |\ell(x)|, \ell \in Y\}$.

Theorem 26

A locally convex space $(V, \{\rho_\alpha\}_{\alpha\in A})$ is a topological vector space

Proof. Let $N(\alpha, \epsilon, z_0)$ be an element of the subbasis on V with the LCS topology.

$$+^{-1}N(\alpha, \epsilon, z_0) = \{(x_0, y_0) : x_0 + y_0 \in N(\alpha, \epsilon, z_0)\}$$

Claim that for any x_0, y_0 $N_{\alpha,x_0,\epsilon-\|x_0+y_0-z_0\|} \times N_{\alpha,y_0,\epsilon-\|x_0+y_0-z_0\|}$ is open and inside the preimage (thus we take the union and we get that the whole preimage is open).

We use the triangle inequality of the seminorms to bound the sum of any two points in this open within ϵ of z_0.

Continuity of multiplication

□

Use the continuity of addition to prove that a LCS is Hausdorff

Definition 46

Two families of seminorms $\{\rho_\alpha\}_{\alpha\in A}, \{d_\beta\}_{\beta\in B}$ on a vector space X are called equivilent, $\{\rho_\alpha\}_{\alpha\in A} \sim \{d_\beta\}_{\beta\in B}$, if they generate the same topology on X

Theorem 27

$\{\rho_\alpha\}_{\alpha\in A} \sim \{d_\beta\}_{\beta\in B}$
$\iff$ all ρ_α are continuous in the topology generated by $\{d_\beta\}$ and likewise all d_β are continuous in the topology generated by $\{\rho_\alpha\}$
$\iff$ $\forall \alpha \in A \exists \beta_1, ..., \beta_n \in B, \exists C > 0, \rho_\alpha(x) \leq C(d_{\beta_1}(x) + ... + d_{\beta_n}(x))$ and $\forall \beta \in B \exists \alpha_1, ..., \alpha_n \in A, \exists C' > 0, d_\beta(x) \leq C'(\rho_{\alpha_1}(x) + ... + \rho_{\alpha_n}(x))$

prove the second iff

Now returning to smooth functions, we have seen that we have a family of seminorms $j \in \mathbb{N}_0, \rho_j(\phi) = max_{x\in[0,1]}|\frac{d^j}{dx^j}\phi(x)|$ as well as a family of norms $d_J(\phi) = \sum_{j=0}^{J} \rho_j(\phi) = \|\phi\|_{C^J}$. These families of seminorms are in fact equivilent in the above sense. So we now give $C^\infty[0, 1]$ the locally convex topology defined by these seminorms.

Theorem 28

Let $(X, \{\rho_\alpha\}), (Y, \{d_\beta\})$ be two locally convex spaces
A linear map $T : X \to Y$ is continuous $\iff$ $\forall \beta \exists \alpha_1, ..., \alpha_n \exists C > 0, \forall x \in X, d_\beta(Tx) \leq C(\rho_{\alpha_1}(x) + ... + \rho_{\alpha_n}(x))$

Proof. □

Lemma 24

A linear function $T : X \to Y$ between Banach spaces is continuous $\iff$ $\exists C$ such that $\|Tx\|_Y \leq C\|x\|_X$

It follows from this that as an automorphism of C^∞ differentiation is continuous.

Definition 47

A family of seminorms $\{\rho_\alpha\}_{\alpha\in A}$ is directed $\iff$ $\forall \alpha_1, ..., \alpha_n \in A \exists \beta \in A \exists C > 0, \rho_{\alpha_1}(x) + ... + \rho_{\alpha_n}(x) < C\rho_\beta(x)$

Theorem 29

Any family of seminorms is equivilent to a directed family

Proof. Let $\{\rho_\alpha\}_{\alpha\in A}$ Define for any $F \in \mathcal{P}^{fin}(A)$ (the set of finite subsets of A)

$$d_F = \sum_{\alpha\in F} \rho_\alpha$$

This is the directed family. $\square$

Theorem 30

For two locally convex spaces X, Y with directed families $\{\rho_\alpha\}, \{d_\beta\}$ a linear map $T : X \to Y$ is continuous $\iff \forall\beta \in B\exists\alpha \in A\exists C > 0, d_\beta(Tx) \leq C\rho_\alpha(x)$

Theorem 31

Hahn-Banach Theorem: Let $(X, \{\rho_\alpha\}_{\alpha\in A})$ a locally convex space and $Y \subseteq X$ a linear subspace

$$\ell : Y \to \mathbb{C} \text{ continuous function } \implies \exists L : X \to \mathbb{C} \text{ a continuous map such that } L|_Y = \ell$$

Proof. The continuity of ℓ implies that $\exists\rho$ some seminorm on X with the property that $\forall y \in Y$ $|\ell(y)| \leq C\rho(y)$ (the bound coming directly from the above lemmas and the fact that $\mathbb{C}$is a LCS with only one (semi)norm given by absolute value).

Then by the previous Hahn-Banach theorem we get the extension $L : X \to \mathbb{C}$ such that $L|_Y = \ell$ and $\forall x \in X$ $|L(x)| \leq C\rho(x)$. $\square$

4.1 Deciding if a topology is that of a LCS

First let $(X, \{\rho_\alpha\}_{\alpha\in A})$ be a locally convex space and let $\mathcal{C} = N_{\rho_{\alpha_1},...,\rho_{\alpha_n},\epsilon} = \{x \in X|\rho_{\alpha_i}(x) < \epsilon, i = 1, ..., n\}$

Definition 48

$\forall\lambda \in \mathbb{C}$ such that $|\lambda| = 1 \implies \lambda N_{\rho_{\alpha_1},...,\rho_{\alpha_n},\epsilon} \subseteq N_{\rho_{\alpha_1},...,\rho_{\alpha_n},\epsilon}$ means that the space is circled

Definition 49

$\mathcal{C}$ is convex $\iff \forall c, y \in \mathcal{C}\forall t \in [0, 1], tx + (1 - t)y \in \mathcal{C}$

Definition 50

C is absorbent $\iff \forall x \in X\exists t > 0, tx \in \mathcal{C}$

Theorem 32

A topological vector space is a LCS $\iff$ there is a neighbourhood basis of $0 \in X$ consiting of circled, convex and absorbent sets (CCA) (all sets in the basis are all three) and X is Hausdorff.

To prove this we will need the following lemmas. Namely we would like to create some seminorms from CCA sets (the if direction has already been shown)

Lemma 25

Minkowski Functional: If $\mathcal{C}$ is a CCS set and $0 \in C$ then $\rho_\mathcal{C}(x) = inf\{t > 0|x \in t\mathcal{C}\}$ is a seminorm

Proof. □

Lemma 26

If $\{\mathcal{C}_\alpha\}_{\alpha\in A}$ is a neighourhood basis of 0 with CCA sets then $\{\rho_{\mathcal{C}_\alpha}\}_{\alpha\in A}$ defines the same topology

We can now prove theorem 32

Proof. □

4.2 Fréchet Spaces

First note that most of what is said, although it is for smooth functions on a compact set of $\mathbb{R}$, will in fact still apply to smooth functions on compact subsets of $\mathbb{R}^n$.

Now recall $C^\infty[a,b]$ the space of smooth functions. This space is a locally convex space (LCS) using the family of seminorms $\rho_j(\phi) = sup_{x\in[a,b]}|\frac{\partial^j}{\partial x^j}\phi(x)|$.
There is another family of seminorms which is also directed and generates the same topology, namely $\|\phi\|_{C^k} = \sum_{j=0}^k \rho_j(\phi)$. This second family makes it clear that we have a countable neighbourhood basis of the origin by restricting $\{N_{k,\epsilon} = \{\phi : \|\phi\|_{C^k} < \epsilon\}\}_{k\in\mathbb{N},\epsilon>0}$ to $\epsilon = \frac{1}{n}$.

The final result about C^∞ is that the LCT is in fact a metric topology, in particular the one given by the following metric

$$d(\phi,\psi) = \sum_{j\geq 1} 2^{-j}\frac{\rho_j(\phi-\psi)}{\rho(\phi-\psi)+1}$$

(or equivilently $\|\cdot\|_{C^k}$ instead of ρ_j).
With respect to this metric topology $C^\infty[a,b]$ is complete.

Proof. That $C^\infty[\mathbb{S}^1]$ is complete in this LC topology □

Lemma 27

For any NLS X the norm topology is equal to the weak topology generated by $\|\cdot - \phi\|$ for $\phi \in X$

Theorem 33

Let X be a LCS then the following are equivilent:

(i) X is metrisable
(ii) $\iff$ there is a countable local basis for the topology at zero
(iii) $\iff$ this is a countable family of seminorms generating the topology

Proof. (i) This is true for any metric space. The balls give a basis on the space so we get a countable neighbourhood basis of 0 from $\{B_{1/n}(0)\}_{n\geq 1}$.

(ii) Take a countable basis of the topology at 0, B and a neighbourhood basis of 0 consisting of convex, circled and absorbing sets B'. We can take a countable subset of B' that forms a neighbourhood basis of 0 because

$$\forall u_n \in B\ \exists u'_n \in B'\ u'_n \subset u_n$$

The minkowski functionals $\{\rho_{u'_n}\}_{n\geq 1}$ give the countable family of seminorms generating the topology. (using the above lemmas) countable family $\implies$ metrisable

□

Definition 51

A Frechet space is a LCS that is metrisable and complete.

Im interested in the following inclusions

$$\text{Banach Spaces} \subset \text{Frechet Spaces} \subset \text{Complete Metric Spaces}$$

Firstly we know that Banach spaces for a category with continuous BLT's, can we say the same about the others? Is there then a notion of one being a subcategory of the other and then is there a notion of the size (relative) of a subcategory in a category.
I feel like in some sense this list of inclusions should be very tight, ie only very pathelogical Frechet spaces are not normable, and likewise for metric spaces?
I want a necissary and sufficient condition on something being Frechet but not Banach, or Metric but not Frechet

In Frechet spaces the theorems such as uniform boundedness, open mapping and Baire category all apply.

Lemma 28

A continuous bijection of Frechet spaces is continuously invertible

Theorem 34

Uniform Boundedness Theorem: Let X, Y be Frechet and $\mathcal{F} : X \to Y$ a family of continuous maps such that for each continuous seminorm ρ on Y and each $x \in X$ $\{\rho(f(x)) : f \in \mathcal{F}\}$ is bounded $\implies$ $\forall \rho$ as above we have that $\exists d_1, ..., d_n \in \{d_\alpha\}_{\alpha \in A}$ (elements of the family of seminorms that defines the topology of X) and $\exists C > 0$ such that $\forall x \in X, \forall f \in \mathcal{F}, \rho(f(x)) \leq C(d_1(x) + ... + d_n(x))$

– Very not precise – Alternatively if a family of continuous functions has a bounded image for each point in its domain then it is equicontinuous

Lemma 29

$F_n : X \to Y$ continuous maps between Frechet spaces such that $\forall x \in X, F_n(x) \to F(x)$ for some F, then F is continuous

4.3 Schwartz Functions

First we set up some notation: Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}$ then define $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ and

$$\partial_x^\alpha = \frac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

We can now define the space of Schwartz functions

Definition 52

$$\mathscr{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}^n, sup|x^\alpha \partial^\beta \phi| < \infty\}$$

These functions are those whose derivatives vanish faster than any polynomial. i.e. The derivative multiplied by a polynomial is always bounded for every polynomial and derivative (in any direction and to any degree).

Lemma 30

$\forall \alpha\beta \in \mathbb{N}^n \exists C_{\alpha,\beta}$ with $|x^\alpha \partial^\beta \phi| \leq C_{\alpha,\beta}$
$\iff \forall N \in \mathbb{N} \forall \beta \in \mathbb{N}^n \exists C_{N,\beta}$ with $|\partial^\beta \phi| \leq \frac{C_{N,\beta}}{(1+|x|^2)^N}$

We define the family of seminorms on $\mathscr{S}(\mathbb{R}^n)$ by $\|\phi\|_{\alpha,\beta} = sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta\phi|$. Note that the family here is doubly indexed by α and β in the appropriate product space of $\mathbb{N}$.

We can give the equivilient directed family of seminorms explicitly also by

$$\|\phi\|_{K,m} = \sum_{|\alpha|\leq K, |\beta|\leq m} \|\phi\|_{\alpha,\beta}$$

Note that Schwartz functions are a Frechet space.

Definition 53

The dual space of $\mathscr{S}$ is called the space of tempered distributions and often denoted with a prime instead of star

$$\mathscr{S}'(\mathbb{R}^n) = (\mathscr{S}(\mathbb{R}^n))^*$$

$u \in \mathscr{S}'(\mathbb{R}^n) \iff \exists K, m \in \mathbb{N}, C > 0$ such that $|u(\phi)| \leq C\|\phi\|_{K,m}$
the dual of infinitely good is finetly bad? What does this bound mean and why is it bad?

We have the following series of inclusions

$$\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$$

Explicitly $\iota : L^1(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$ is given by $f \mapsto (f)(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$

Prove this ι is injective

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ then we have that

$$\int |\phi x| dx \leq \int \frac{< x >^{2(n+1)}}{< x >^{n+1}} |\phi x| dx \leq \|\phi\|_{n+1,0} \int \frac{1}{< x >^{n+1}} dx$$

Where $< x > = (1+|x|^2)^{\frac{1}{2}}$ is the "Japanese bracket" and has a convergent integral on $\mathbb{R}^n$. Thus the final thing on the right is finite and we have shown that ϕ is integrable. □

The above proves that fact that

Lemma 31

There is an inclusion of $\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$ given by $\iota(\phi)(\psi) = \int \phi(x)\psi(x)dx$.
Moreover this embedding is a continuous map in the weak star topology

Lemma 32

$$\{u_i\} \subseteq \mathscr{S}' \qquad u_i \to u \iff \forall \phi \in \mathscr{S} \qquad u_n(\phi) \to u(\phi)$$

Prove the backwards implication. Forward implication was proved in class (the only use of the weak star topology that we made)

4.3.1 Operations on Tempered Distributions

For a continuous funciton $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ we want to be able to extend it to a $\tilde{T} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ such that $\tilde{T}|_S = T$ i.e. that $\tilde{T}\iota(\phi) = \iota T(\phi)$.

Differentiation Operator: Define an operator $T = \frac{d}{dx} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. Now we define

$$\begin{aligned}\iota(T(\phi))(\psi) &= \int_{\mathbb{R}} T(\phi)(x)\psi(x)dx \\ &= \int \phi(-\frac{d}{dx}\psi)dx \text{ integrate by parts and vanishes on the boundary} \\ &= \iota(\phi)(\frac{d}{dx}\psi)\end{aligned}$$

So if we define $(\tilde{T}u)(\psi) = u(-\frac{d}{dx}\psi)$ we get that $\tilde{T}(\iota(\phi))) = \iota(T(\phi))$.

One can check that this is a well definied BLT $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$. (The continuity of this map follows from the fact that it is the adjoint of a continuous map). In particular we have that $T = \frac{d}{dx}$ and $\tilde{T} = L'$ where $L : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), \psi \mapsto -\frac{d}{dx}\psi$ since

$$\tilde{T}(u)(\psi) = L'(u)(\psi) = u(L\psi) = u(-\frac{d}{dx}\psi)$$

In general we can define or show that for $u \in \mathscr{S}'(\mathbb{R}^n)$

$$\frac{\partial^\alpha}{\partial x^\alpha}(u)(\phi) = u((-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\phi)$$

Multiplication By a Smooth Function Let $a \in C^\infty_\infty(\mathbb{R}^n)$ (smooth and bounded functions) and define $M_a : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by $\phi \mapsto a\phi$ we can extend this to $\tilde{M}_a : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by

$$\begin{aligned}\iota(M_a(\phi))(\psi) &= \int (M_a\phi)\psi \\ &= \int (a\phi)\psi \\ &= \int \phi(a\psi) \\ &= (\iota(\phi))(M_a\psi)\end{aligned}$$

So we let $\tilde{M}_a(u)(\phi) = u(M_a\phi)$. We can also notice that in this case $\tilde{M}_a = M_a^*$ (so inparticular it is continuous)

Convolutions We have the function $\phi \star \psi(x) = \int \phi(x-y)\psi(y)dy$ for any two $\phi, \psi \in \mathscr{S}(\mathbb{R}^n)$ so we can define a function $C_\phi : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by $\psi \mapsto \phi \star \psi$. Again we aim to extend it

$$\begin{aligned}\iota(C_\phi(\gamma))(\psi) &= \int C_\phi(\gamma)(x)\psi(x)dx \\ &= \int (\phi \star \gamma)(x)\psi(x)dx \\ &= \int\int \phi(x-y)\gamma(y)dy\psi(x)dx \\ &= \int \gamma(y) \int \phi(-(y-x))\psi(x)dxdy\end{aligned}$$

Which suggests that $\tilde{C}_\phi u(\psi) = u(C_{\phi(-\cdot)}\psi)$ (multiply the input to ϕ by negative one then proceed.)

> Prove C_ϕ is continuous as a function between Schwartz functions. Prove that $\tilde{C}_\phi \in C^\infty$

Dirac Delta For a given $x_0 \in \mathbb{R}^n$ we can define the dirac delta as $\delta_{x_0} : \mathscr{S} \to \mathbb{C}$ that $\phi \mapsto \phi(x_0)$. We can verify this is a distribution becuase

$$|\delta_{x_0}(\phi)| = |\phi(x_0)| \le sup|\phi(x)| = sup|a^0(\frac{d}{dx})^0\phi(x)| = \|\phi\|_{0,0}$$

And so we have shown that it is a distribution.

We can also define $\delta'_{x_0}(\phi) = -\frac{d}{dx}\phi|_{x=x_0}$ the derivative of the dirac delta.

Lemma 33

$\mathscr{S}(\mathbb{R}^n)$ is dense in $\mathscr{S}'(\mathbb{R}^n)$

In particular if an extension is given as above it is unique.

5 Bounded Operators, Spectrum and Compactness

5.1 Topologies and Adjoints

For the space of bounded linear transformations from one Banach space to another $\mathcal{L}(X,Y)$ we have the norm topology already defined as well as the weak topology from its dual, we will also here give two more topologies that naturally arise on such spaces.

Definition 54

For an $x \in X$ the evalution map $E_x : \mathcal{L}(X,Y) \to Y$ sending $A \mapsto Ax$ generates a family of functions $\mathcal{F} = \{E_x | x \in X\}$. The weak topology on $\mathcal{L}(X,Y)$ given by $\mathcal{F}$ is the Strong Operator Topology (SOT).

For $x_1, ..., x_n \in X, \epsilon > 0$ the set $\{A \in \mathcal{L}(X,Y) : \|Ax_i\|_Y < \epsilon, \forall i\}$ is open in the SOT

Lemma 34

$A_n \to A$ in the SOT $\iff \forall x \in X, A_n x \to Ax \in Y$

Convergence in the strong operator topology is pointwise convergence

Proof. $A_n \xrightarrow{SOT} A \iff A_n - A \xrightarrow{SOT} 0$

$$\iff \forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N \quad A_n - A \in B_\epsilon(0) = \{\phi \in \mathcal{L}(X,Y) : \|\phi x\| < \epsilon\}$$

From the definition of open neighbourhoods in the SOT. $\iff \forall x \in X \quad \|A_n x - Ax\| < \epsilon$ which is exactly $A_n x \to Ax$. □

Lemma 35

For a given $B \in \mathcal{L}(X,Z)$ the function $M_B : \mathcal{L}(X,Y) \to \mathcal{L}(X,Z), A \mapsto M_B A = BA$ is continuous in the SOT

Proof. Let $\mathcal{U} = \{C \in \mathcal{L}(X,Z) : \|Cx_i\|_Z, \quad i = 1, ..., n\}$ be a generic open set. Then we take its preimage $M_B^{-1}(\mathcal{U}) = \{A : \|BAx_i\|_Z < \epsilon, \quad i = 1, ..., n\}$. Then

$$\|BAx_i\|_Z \leq \|B\|_{\mathcal{L}(X,Y)} \|Ax_i\|$$

Because the operator norm is the smallest such constant that bounds B in this way (standard manoeuvre). Thus

$$\{A : \|Ax_i\|_Y < \frac{\epsilon}{\|B\|}\} \subseteq M_B^{-1}(\mathcal{U})$$

So we have shown that around any point in $M^{-1}(\mathcal{U})$ we can put an open ball and so its open. □

Definition 55

Taking the family of functions $\mathcal{F} = \{\lambda \circ E_x\}_{x \in X, \lambda \in Y^*}$ and giving X the weak topology generated by $\mathcal{F}$ is giving the Weak Operator Topology (WOT)

Again we can characterise open sets in the WOT as sets of the form $\{A : |\ell_i(A(x_j))| < \epsilon, i = 1, ..., k, j = 1, ..., p\}$ where $\ell_1, ..., \ell_k \in Y^*, \epsilon > 0$ and $x_1, ..., x_p \in X$ are given.

Prove that M_B is continuous in the WOT too

Lemma 36

$A_n \to A$ in the WOT $\iff \forall \lambda \in Y^*, \forall x \in X, \lambda(A_n x) \to \lambda(Ax)$

We have the following topologies then

$$WOT \subset SOT \subset \text{Norm Topology}$$

If Y and Z are infinite dimensional spaces then $\mathcal{L}(X,Y) \times \mathcal{L}(Y,Z) \to \mathcal{L}(X,Z), (A,B) \mapsto BA$ is continuous in the norm topology but not in the SOT or WOT

Comparing Convergence: Consider the following three families of maps

- $N_n : \ell_p \to \ell_p, \qquad x \mapsto \frac{1}{n}x$
- $L_n : \ell_p \to \ell_p, \qquad (x_1, x_2, ...) \mapsto (x_{n+1}, x_{n+2}, ...)$ (left shift)
- $R_n : \ell_p \to \ell_p, \qquad (x_1, x_2, ...) \mapsto (0, 0, ..., x_1, x_2, ...)$ i.e. adding n zeros to the front (right shift)

We have that $N_n \to 0$ in norm, $L_n \to 0$ in SOT and $R_n \to 0$ in WOT.

Proof.

$$\|N_n - 0\| = \sup_{\|x\|=1} \|A_n x\| = \sup_{\|x\|=1} \|\frac{1}{n}x\| = \frac{1}{n} \to 0$$

- -

First let $x \in \ell^p$ then we can see pointwise convergence

$$\|(L_n - 0)x\| = (\sum_{i \geq 1} |x_{n+i}|^p)^{\frac{1}{p}} = (\sum_{i \geq n} |x_i|^p)^{\frac{1}{p}} \to 0$$

Because the tails of convergent sums always go to zero.

- -

First we note that $\ell^q = (\ell^p)^*$ so we can take $y \in \ell^q, x \in \ell^p$ and then we can see that

$$|y(R_n x)| = |\sum_{i \geq 1} \bar{y}_i (R_n x)_i| = |\sum_{i \geq 1} \bar{y}_{n+i} x_i| \leq \|L_n y\|_{\ell^q} \|x\|_{\ell}^p \to 0$$

Where the last bound follows because

$$\sum \bar{y}_{n+i} x_i = (L_n y, x)$$

A psuedo innerproduct defined by summing over the multiplied entries. Then we can approximate using Holders inequality because p and q are conjugate. Finally we have alread shown $L_n y \to 0$ for any y above and so the final convergence is shown.

□

5.1.1 Ajoints

Recall the spectral theorem on matricies states that for any symetric matrix there is some orthogonal matrix that will diagonalise it, moreover the diagonal entries will be the eigen values of the original matrix.

This is the kind of characterisation we want to be able to build up to for operators.

We first define the adjoint of a bounded linear transformation

Definition 56

For a BLT $T : X \to Y$ its adjoint is the map $T' : Y^* \to X^*$ sending $\lambda \mapsto \lambda \circ T$

In particular we have an isometry $\Phi : \mathcal{L}(X,Y) \to \mathcal{L}(Y^*,X^*)$ where $T \mapsto T'$.
If X and Y are reflexive we have that Φ is a bijection and

$$T_n \xrightarrow{s} T \implies T_n' \xrightarrow{w} T'$$

Proof.

$$\begin{aligned}
\|T'\|_{\mathcal{L}(X^*,Y^*)} &= \sup_{\|\ell\|_{Y^*}\le 1} \|T'\ell\|_{X^*} \\
&= \sup_{\|\ell\|_{Y^*}\le 1} \left(\sup_{\|x\|_X\le 1} \|(T'\ell)(x)\|\right) \\
&= \sup_{\|\ell\|_{Y^*}\le 1} \sup_{\|x\|_X\le 1} |\ell(T(x))| \\
&= \sup_{\|x\|_X\le 1} \sup_{\|\ell\|_{Y^*}\le 1} |\ell(T(x))| \\
\star &= \sup_{\|x\|_X\le 1} \|T(x)\|_Y \\
&= \|T\|_{\mathcal{L}(X,Y)}
\end{aligned}$$

Where the line $\star$ follows from $\sup_{\|\ell\|_{Y^*}\le 1} |\ell(T(x))| \ge \|T(x)\|$ because by Hahn-Banach there is a functional of norm 1 that takes the value $\|T(x)\|$ at that point. The reverse bound is $\sup_{\|\ell\|_{Y^*}\le 1} |\ell(T(x))| \le \sup_{\|\ell\|_{Y^*}\le 1} \|\ell\|\|T(x)\| = \|T(x)\|$.

Note that isometry implies injective.

Now assuming that X and Y are reflexive then we get that

$$\tilde{T} \in \mathcal{L}(Y^*,X^*) \implies \tilde{T}' \in \mathcal{L}(X^{**},Y^{**}) = \mathcal{L}(X,Y)$$

And so we have shown that $(\tilde{T}')' = \tilde{T}$. □

Why does the final line show that the map is a bijection? I only see that we have shown we will land in the correct space when adjointing twice but not that it will literally be the same element

For a Hilbert space and a map $T \in \mathcal{L}(\mathcal{H},\mathcal{H}) = \mathcal{L}(\mathcal{H})$ we also get that $(x,Ty) = (T^*x,y)$ for some map T^* that we will also call the adjoint. This is a unique map by the Riesz Representation theorem.

T^* and T' are related by the Riesz map $C : \mathcal{H} \to \mathcal{H}^*, x \mapsto (x,-)$ and the following relation, $C^{-1} \circ T' \circ C = T^*$. Where $C : \mathcal{H} \to \mathcal{H}^*, C(x)(y) = (x,y)$.

Lemma 37

$$\|T\| = \|T^*\|$$

We should be careful however because $(aT)^* = \bar{a}T^*$

Lemma 38

- $\forall S,T : \mathcal{H} \to \mathcal{H}$ BLT's, we have that $(ST)^* = T^*S^*$ and $(ST)' = T'S'$
- $(T^*)^* = T$
- For an invertible T; $(T^{-1})^* = (T^*)^{-1}$
- $\|T^*T\| = \|T\|^2$

Proof. All simple computations except for the fourth. We will bound each by the other to show equality:

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|\|T\| = \|T\|^2$$

Where the inequality follows from the definition of the operator norm and we apply the lemma above (that $T \to T^*$ is an isometry) for the equality.

The other bound is given by

$$\begin{aligned}
\|T^*T\| &= \sup_{\|x\|=1} \|T^*T(x)\| \\
&= \sup_{\|x\|=1} \|T^*T(x)\|\|x\| \\
\star &\geq \sup_{\|x\|=1} (x, T^*Tx) \\
&= \sup_{\|x\|=1} (Tx, Tx) \\
&= \sup_{\|x\|=1} \|Tx\|^2 \\
&= \|T\|^2
\end{aligned}$$

$\star$ is Cauchy-Schwartz.

Note that we subtle use of the Riesz Representation theorem here, in the first inequality thinking of T^* as an element of $\mathcal{L}(\mathcal{H}^*)$ and in the second using it as an element of $\mathcal{L}(\mathcal{H})$. $\square$

$* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), T \mapsto T^*$ is continuous in the norm topology and the weak topology but not the strong operator topology when $\mathcal{H}$ is infinite dimensional.

In general we can say that for a map $T : X \to Y$ that is continuous then it remains continuous if X is given a finer topology or Y is given a coarser topology. But if both topologies are made coarser or both finer, nothing can be said in general.

5.2 Projections

Definition 57

A map $P \in \mathcal{L}(\mathcal{H})$ is a projection if $P^2 = P$. It is further an orthogonal projection if $P = P^*$ (self adjoint)

Lemma 39

For any orthogonal projection P we have that $\mathcal{H} = ker(P) \oplus Im(P)$. Moreover these two subpaces are closed and orthogonal

Proof. The kernel of a continuous map is always closed because the preimage of closed sets are closed and

$$kerP = \{P^{-1}(0)\}$$

Next let $x \in Im(p)$ then for some $y \in \mathcal{H}$ we have that $x = Py$ by orthogonality then $Px = P^2y = Py$ so $Px = x$ i.e.

$$x \in Im(P) \iff x \in ker(I - P)$$

Where I is the identity map. Again $I - P$ is continuous, so has closed kernel and therefore P has a closed image.
Clearly $\forall x \in \mathcal{H} \quad x = Px - Px + x = Px + (I - P)x$ so $\mathcal{H} = ker(P) \oplus Im(P)$, moreover for $x \in ker(P)$

and $P(y) \in Im(P)$:

$$\begin{aligned}(x, Py) &= (P^* x, y)\\ &= (Px, y)\\ &= (0, y)\\ &= 0\end{aligned}$$

□

A final note is that orthogonal projections are in one to one corrospondence with closed subspaces because for $\mathcal{H} = M \oplus M^\perp$ we have an orthogonal projection definied by $P_M, x = m + m^\perp \mapsto m$.

5.3 Spectrum and Resolvent

A notational convenience adopted henceforth will be that for operators A, I where I is the identity and a scalar λ we will write $A - \lambda I = A - \lambda$. i.e. we will omit the identity map when it is clear that it must be there (to add scalars to operators)

Definition 58

Let $A \in \mathcal{L}(X)$ where X is a Banach space. Then we define

- The resolvent set $\rho(A) = \{\lambda \in \mathbb{C} | A - \lambda \text{ is invertible } \}$
- The spectrum of A $\sigma(A) = \mathbb{C} \setminus \rho(A)$
- For a $\lambda \in \sigma(A)$ if $\lambda - A$ is non-injective we say λ is in the point spectrum of A. Moreover $\exists x \in X, Ax = \lambda x$ so we call x an eigenvector and λ its eigenvalue
- If $\lambda - A$ is injective but $Im(\lambda - A)$ is not dense then we say λ is in the residual spectrum

Lemma 40

$\rho(A)$ is open

Lemma 41

For B a BLT $\{I + B : \|B\| \leq 1\} \subseteq$ invertible operators

Definition 59

The resolvent function is defined as $R_\lambda(A) = (A - \lambda)^{-1}$ from the appropriate domain.

Definition 60

A function $f : \Omega \to X$, where Ω is a domain in $\mathbb{C}$and X is any Banach space, is (weakly) analytic

$$\iff \quad \forall \lambda \in X^* \quad \lambda \circ f : D \to \mathbb{C} \text{ is analytic}$$

Lemma 42

$R_\lambda(A)$ is an analytic function (in λ) on the domain $\rho(A)$

Proof. □

Lemma 43

We define $r(A) = sup\{|\lambda| : \lambda \in \sigma(A)\}$ the spectral radius. We then have the following facts

- $r(A) \leq \|A\|$
- $r(A) = lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}$ (in particular this limit exists)
- $A = A^* \implies r(A) = \|A\|$

Proof. Proof of r(A)< $\|A\|$ relies on previous proof that I am yet to understand.

Show that the limit exists $\lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}$

We show that the spectral radius is less then and greater than and hence equal to this limit. First recall the fact that any complex power series $\sum_{n\geq 1} a_n z^n$ has radius of convergence given by $(\lim_{n\to\infty} sup|a_n|^n)^{-1}$. As before we are going to write a power series that on its radius of convergence gives the inverse of $\lambda - A$. So consider $\star\frac{1}{\lambda}(I + \sum_{k\geq 1}\lambda^{-k}A^k)$. This is absolutely summable for

$$|\lambda| > \lim_{k\to\infty} sup\|A^k\|^{\frac{1}{k}} = \lim_{k\to\infty}\|A^k\|^{\frac{1}{k}}$$

We see that for such a λ our series is

$$\frac{1}{\lambda}(I + \sum_{k\geq 1}\lambda^{-k}A^k) = (\lambda - A)^{-1}$$

Thus $r(A) \leq \lim_{k\to\infty}\|A^k\|^{\frac{1}{k}}$.

Now to show equality we assume for a contradiction that $r(A) < \lim_{k\to\infty}\|A^k\|^{\frac{1}{k}}$.

By the analyticity of $R_\lambda(A)$ in the resolvent we have that $\star$ converges for

$$|\frac{1}{\lambda}| \leq \frac{1}{\lim_{k\to\infty}\|A^k\|^{\frac{1}{k}}} + \epsilon$$

This however is a contradiction because then for $|\frac{1}{\lambda}| = \frac{1}{\lim_{k\to\infty}\|A^k\|^{\frac{1}{k}}} + \frac{\epsilon}{2}$ the series would diverge. □

Lemma 44

$sup\{|\lambda| : \lambda \in \rho(A)\} = \|A\|_{\mathcal{L}(X)}$

Prove this or find a proof. What does this mean in light of the above fact about self adjoint A and the radius of the spectrum?

Lemma 45

$\forall\mu, \lambda \in \rho(A), R_\mu(A)R_\lambda(A) = R_\lambda(A)R_\mu(A)$. i.e. the inverse of two different translates of A will commute.

Proof. □

Recall the adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{L}^*(\mathcal{H})$ has the property $(T^*x, y) = (x, Ty)$.

Definition 61

An operator is called self adjoint if $T = T^*$

Definition 62

We call λ an eigenvalue of an operator A if $\lambda \in \sigma(A)$ and $\exists \phi \in X$ such that $A\phi = \lambda\phi$

Definition 63

$\lambda \in \sigma_{Res}(A) \iff \lambda$ (the residual spectrum) is not an eigenvalue, $ker(A-\lambda) = \{0\}$ and $im(A-\lambda)$ is not dense

Theorem 35

Let $T \in \mathcal{L}(\mathcal{H})$ be self adjoint $\implies$

- $\sigma(T) \subseteq \mathbb{R}$ i.e. the spectrum is real
- λ_1, λ_2 eigen values of $T\phi_1 = \lambda_1\phi_1, T\phi_2 = \lambda_2\phi_2 \implies (\phi_1, \phi_2) = 0$
  i.e. Eigenvectors corrosponding to unique eigenvalues are orthogonal
- There is no residual spectrum i.e. (Below are two equivilent conditions for an empty residual spectrum)

$$\forall \lambda \in \mathbb{R}, Im(T-\lambda)^{\perp} = ker(T-\lambda)$$

$$\lambda \in \sigma(A) \implies ker(\lambda - A) \neq \emptyset \vee Im(\lambda - A) \text{ is dense}$$

5.4 Compact Operators

Definition 64

A BLT $T : X \to Y$ is compact
$\iff$ for all bounded sequences $(x_n) \subset X$ (Tx_n) is precompact
$\iff$ S bounded implies $T(S)$ is precompact

Continuous Integral Kernel Let $T : C[0,1] \to C[0,1]$ be the function defined by a fixed $K \in C([0,1]^2)$:

$$T(f)(x) = \int_{[0,1]} K(x,y)f(y)dy$$

We can show that this is a compact operator as follows

Proof. Take a sequence $\{f_n\} \subseteq C[0,1]$ that is bounded i.e. $\exists C > 0 \; \forall n \in \mathbb{N} \; \|f_n\| < C$. Then applying T we get that

$$\begin{aligned} |T(f_n x) - T(f_n x')| &= |\int_{[0,1]} K(x,y)f_n(y)dy - \int_{[0,1]} K(x',y)f_n(y)dy| \\ &\leq \int_{[0,1]} |K(x,y) - K(x',y)||f_n(y)|dy \\ \star &\leq \epsilon' \int_{[0,1]} |f(y)|dy \\ &\leq C\epsilon' \end{aligned}$$

We justify the bound in $\star$ because K is continuous, i.e. $\forall \epsilon' \; \exists \delta \; |x-x'| < \delta \implies |K(x,y)-K(x',y)| < \epsilon$. So we have now that

$$\forall x \in [0,1] \; \forall \epsilon \; \exists \delta \; \forall n \in \mathbb{N} \; |x-x'| < \delta \implies |Tfx - Tfx'| < \epsilon$$

Because we can take the epsilon to be $\frac{1}{C}\epsilon'$ and the δ where these are the constants supplied by the definition of K continuity. This is exactly the statement that $\{Tf_n\}$ is an equicontinuous family of functions. So applying Arzelia-Ascoli, to a family of uniformly bounded equicontinuous functions on $[0, 1]$, we get that there is a convergent subsequence just as required. □

Lemma 46

If S or T is compact then $T \circ S$ is compact

Proof. Follows immediately from two facts: Continuous images of bounded sequences are bounded and continuous images of convergent sequences are convergent. □

Lemma 47

$T : X \to Y$ compact and $(x_n) \to x$ weakly $\implies$ $(Tx_n) \to Tx$ in the norm

Proof. S here is strongly continuous and therefore weakly continuous. Thus we have that $T(x_n) \to T(x)$ weakly.

Next by the uniform boundedness principle we have that $\{x_n\}$ is bounded. We get this by thinking of $x_n \in (X^*)^*$ and then considering the set $\{\tilde{x}_n(\ell)\} = \{\ell(x_n)\}$ which is bounded for all $\ell \in X^*$ (because it is weakly convergent).

Thus we have the image of a bounded sequence under a compact operator and can take a convergent subsequence $\{T(x_{n_k})\}_{k\geq 1}$ (convergent in the norm). We know that $T(x_{n_k}) \to Tx$ because it converges in the norm to something and which implies it converges weakly to the same thing, however limits are unique in a Hausdorff topology (and the weak topology is Hausdorff) so this limit must be the same as the weak limit assumed, namely Tx.

Now assume for a contradiction that $Tx_n \not\to Tx$: This implies that $\exists \epsilon > 0$ and a subsequence x_{n_j} such that

$$\|Tx_{n_j} - Tx\| > \epsilon$$

By compactness of T again we can take a strongly convergent subsubsequence of Tx_{n_j} converging to Tx, thus a contradiction.

□

Definition 65

An operator A has finite rank $\iff$ $im(A)$ has finite dimension

Theorem 36

- The space of compact operators $Com(X) \subseteq \mathcal{L}(X)$ is closed in the operator norm topology
- $\mathcal{H}$ is a seperable Hilbert space and $K \in Com(\mathcal{H})$
  $\implies$ $\exists F_n \in \mathcal{L}(\mathcal{H})$ with finite rank such that $\|F_n - K\| \to 0$ as $n \to \infty$
  i.e. $Comp(\mathcal{H}) = \overline{\{F : F \text{ has finite rank }\}}$ (norm closure)

Proof.

□

Theorem 37

Fredholm Alternative: For a (the) seperable hilbert space $\mathcal{H}$ and a compact operator K: We have the following formulations of the theorem

$$I-K \text{ is invertible } \iff I-K \text{ is injective}$$

$$I-K \text{ is not invertible } \iff 1 \text{ is in the point spectrum of } A$$
$$\iff \exists x \in \mathcal{H}, Kx = x$$

Proof. □

A corrollorary of this is that $\sigma(K) \setminus \{0\} = \{$ eigenvalues of K $\}$

Theorem 38

Analytic Fredholm Theorem: For $z \in \Omega \subseteq \mathbb{C}$ where Ω is a domain and $K(z)$ an analytic (as a function from $\Omega \to Comp(\mathcal{H})$) and compact function on $\mathcal{H}$ a seperable Hilbert space we have one of the following two things:

- $I - K(z)$ is not invertible (on all of Ω)
- $I - K(z)$ is invertible off of a discrete set $D \subseteq \Omega$ i.e. $(I - K(z))^{-1}$ is meromorphic. Further any singularities are poles (not essential).

  The singularities will be in D.

Proof. □

Theorem 39

Riesz-Schander: $T \in Comp(\mathcal{H})$ where $\mathcal{H}$ is a seperable Hilbert space $\implies$

$\sigma(T)$ is discrete away from zero and $\forall \lambda \in \sigma(T) \setminus \{0\}, 0 < dimker(T - \lambda) < \infty$

Proof. □

Theorem 40

Spectral Theorem for Self Adjoint Compact Operators:
$T \in Comp(\mathcal{H})$ a compact operator on a seperable hilbert space such that $T = T^*$ (self adjoint) $\implies \exists (\phi_i)_{i\geq 1}$ a countable ONB of $\mathcal{H}$ consisting of eigenfunctions of T ($T\phi_i = \lambda_i \phi_i$).
Moreover we can order the eigenvalues such that $lim_{n\to\infty} |\lambda_n| = 0$

Proof. □

5.5 Functional Calculus

Lemma 48

For a polynomial P $P(\sigma(A)) = \sigma(P(A))$

Let $A \in \mathcal{L}(\mathcal{H})$ a self adjoint operator. We have the following map $\phi : Poly \to \mathcal{L}(\mathcal{H})$ such that $\phi(P) \mapsto P(A)$. More specifically $\sum a_i x^i \mapsto \sum a_i A^i$. We can uniquely extend ϕ to a continuous map $\tilde{\phi} : C(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:

Lemma 49

- $\forall f \in C(\sigma(A)), \|\tilde{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_\infty$
- $\sigma(\tilde{\phi}(f)) = f(\sigma(A))$ (spectral mapping property)
- $\psi \in \mathcal{H}, A\psi = \lambda\psi \implies \tilde{\phi}(f)(\psi) = f(\lambda)(\psi)$
- $(\tilde{\psi}(f))^* = \tilde{\phi}(\overline{f})$

Note that both ϕ and $\tilde{\phi}$ are $\star$-Algebra Homomorphisms, meaning they have the following properties (for functions in their respective domains):

- $\phi(P+Q) = \phi P + \phi Q$
- $\phi(PQ) = \phi(P)\phi(Q)$
- $\phi(1) = A$
- $\phi(cP) = c\phi(P)$
- $(\phi(P))^* = \phi(\bar{P})$

Proof. □

We have now made sense of functions of bounded self adjoint operators. This is one of the two things that the spectral theory allowed us to do for compact SA operators.

5.5.1 Unitary Equivilence To Multiplication

So the functional calculus is a version of the spectral theorem for bounded self adjoint operators, there is another formulation that is in terms of unitary equivilience to multiplication (operator).

Theorem 41

$A \in \mathcal{L}(\mathcal{H}), A = A^*$ then there is some measure space (M, μ) such that $\mu(M) < \infty$, a unitary isomorphism $U : \mathcal{H} \to L^2(M, \mu)$ and a bounded map $F : M \to \mathbb{R}$ such that $UAU^{-1} = F$

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